Tikrit university
College of Engineering
Mechanical Engineering Department

Lectures on Numerical Analysis

Chapter 5

Solving the Ordinary Differential Equations

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What is Euler’s method?

Euler’s method is a numerical technique to solve ordinary differential equations of the form

$$\frac{dy}{dx} = f(x, y), y(0) = y_0$$

So only first order ordinary differential equations can be solved by using Euler’s method. How does one write a first order differential equation in the above form?

Example

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, y(0) = 5 \qquad \frac{dy}{dx} = f(x, y), \ y(0) = y_0$$

is rewritten as

$$\frac{dy}{dx} = 1.3e^{-x} - 2y, y(0) = 5$$

In this case

$$f(x, y) = 1.3e^{-x} - 2y$$

Euler's Method

Derivation of Euler's method

At $x= 0$ we are given the value of $y = y_0$. Let us call $x= 0$ as x_0 . Now since we know the slope of y with respect to x that is, $f(x, y)$ then at (x, x_0), the slope is $f(x_0, y_0)$

$$\text{Slope} = \frac{Rise}{Run} = \frac{y_1 - y_0}{x_1 - x_0}$$

$$= f(x_0, y_0)$$

$$y_1 = y_0 + f(x_0, y_0)(x_1 - x_0)$$

$$= y_0 + f(x_0, y_0)h$$

$$y_{i+1} = y_i + f(x_i, y_i)h \quad h = x_{i+1} - x_i$$

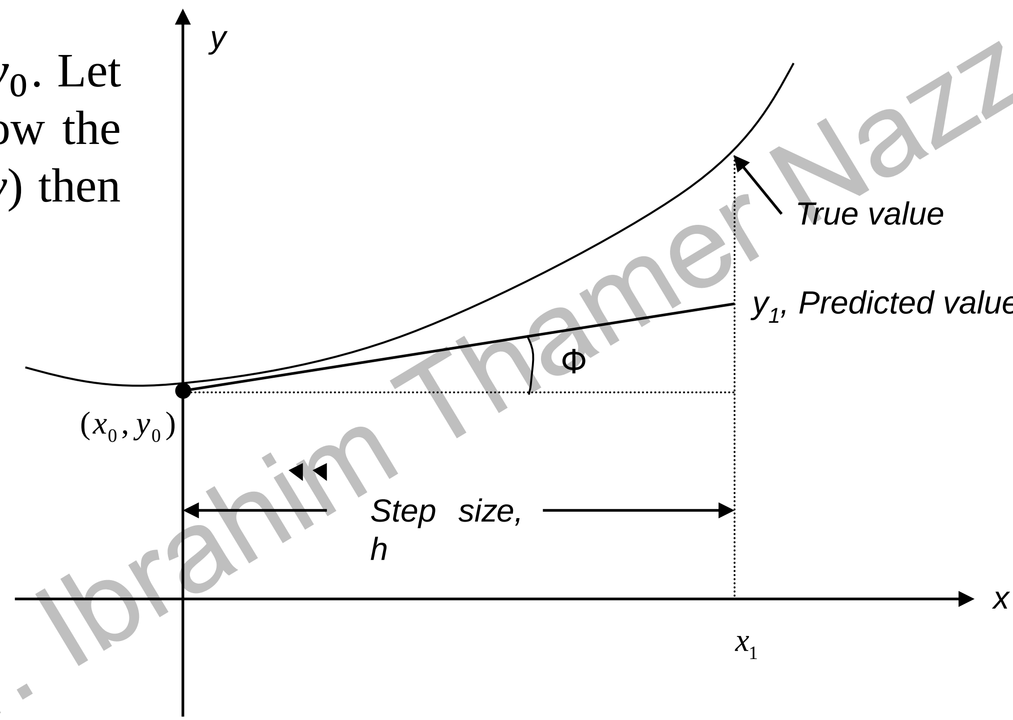

Figure Graphical interpretation of the first step of Euler's method

This formula is known as Euler's method

Example A ball at 1200K is allowed to cool down in air at an ambient temperature of 300K. Assuming heat is lost only due to radiation, the differential equation for the temperature of the ball is given by

$$\frac{d\theta}{dt} = -2.2067\times10^{-12}\left(\theta^4 - 81\times10^8\right), \theta(0) = 1200K$$

Find the temperature at t= 480 seconds using Euler's method. Assume a step size of h= 240 seconds

Solution

Step 1: $$\frac{d\theta}{dt} = -2.2067\times10^{-12}\left(\theta^4 - 81\times10^8\right)$$

$$f(t,\theta) = -2.2067\times10^{-12}\left(\theta^4 - 81\times10^8\right)$$

$$\theta_{i+1} = \theta_i + f(t_i, \theta_i)h$$

$$\theta_1 = \theta_0 + f(t_0, \theta_0)h$$

$$= 1200 + f(0,1200)240$$

$$= 1200 + \left(-2.2067\times10^{-12}\left(1200^4 - 81\times10^8\right)\right)240$$

$$= 1200 + (-4.5579)240$$

$$= 106.09K$$

θ_1 is the approximate temperature at $t = t_1 = t_0 + h = 0 + 240 = 240$

$$\theta(240) \approx \theta_1 = 106.09K$$

Step 2: For $i=1, \quad t_1=240, \quad \theta_1=106.09$

$$\theta_2=\theta_1+f(t_1,\theta_1)h$$
$$=106.09+f(240,106.09)240$$
$$=106.09+\left(-2.2067\times10^{-12}\left(106.09^4-81\times10^8\right)\right)240$$
$$=106.09+(0.017595)240$$
$$=110.32K$$

θ_2 is the approximate temperature at $t=t_2=t_1+h=240+240=480$

$$\theta(480)\approx\theta_2=110.32K$$

The exact solution of the ordinary differential equation is given by the solution of a non-linear equation as

$$0.92593\ln\frac{\theta-300}{\theta+300}-1.8519\tan^{-1}(0.00333\theta)=-0.22067\times10^{-3}t-2.9282$$

The solution to this nonlinear equation at t=480 seconds is $\theta(480)=647.57K$

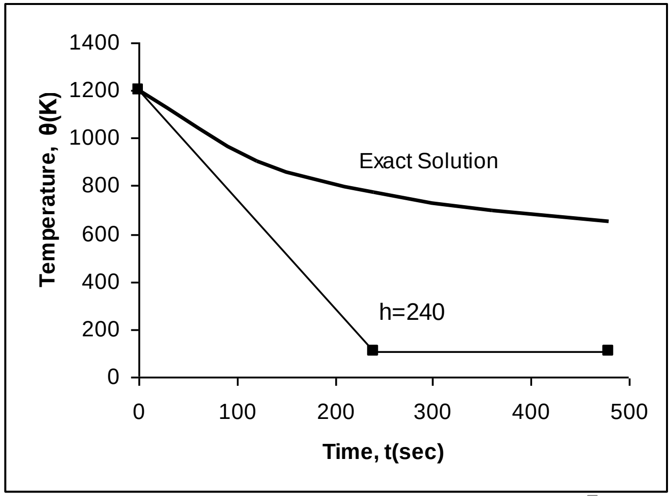

Figure Comparing exact and Euler's method

Modified Euler's Method

The modified Euler method is a modification of Euler's explicit method. As discussed, the main assumption in Euler's method is that in step the derivative (slope) between points (x_i, y_i) and $(x_{i+1}, y_{i+1})$is constant and equal to the derivative (slope) of *y(x)* at point (x_i, y_i). This assumption is the main source of error. In the modified Euler method the slope used for calculating the value of y_{i+1} is modified to include the effect that the slope changes within the subinterval. The slope used in the modified Euler method is the average of the slope at the beginning of the interval and an estimate of the slope at the end of the interval. The slope at the beginning is given by:

$$\left.\frac{dy}{dx}\right|_{x=x_i} = f(x_i, y_i)$$

The estimate of the slope at the end of the interval is determined by first calculating an approximate value for y_{i+1} written as using Euler's explicit method:

$$y_{i+1} = y_i + f(x_i, y_i)h$$

and then estimating the slope at the end of the interval by substituting the point (x_{i+1}, y_{i+1}) in the equation for $\frac{dy}{dx}$

$$\left.\frac{dy}{dx}\right|_{x=x_{i+1}} = f(x_{i+1}, y_{i+1})$$

The modified Euler method is summarized in the following steps.

1. Given a solution at point (x_i, y_i) calculate the next value of the independent variable:
 $x_{i+1} = x_i + \text{h}$
2. Calculate $f(x_i, y_i)$.
3. Estimate y_{i+1} using Euler's method: $y_{i+1} = y_i + f(x_i, y_i)h$
4. Calculate (x_{i+1}, y_{i+1}).
5. Calculate the numerical solution at x_{i+1}: $y_{i+1} = y_i + \frac{h}{2}[f(x_i, y_i) + f(x_{i+1}, y_{i+1})]$

Example:- Use the modified Euler method to solve the ODE

$\frac{dy}{dx} = x^2 + 4x - \frac{1}{2}y$ for $x = 0$ *to* 0.1 with the initial condition $y(0) = 4$. Using $h = 0.05$.

Solution

Given $\frac{dy}{dx} = x^2 + 4x - \frac{1}{2}y$, *y(0) = 4, h = 0.05 and find y(0.0.5) and y(0.1)* . *Here* $x_0 = 0, y_0 = 4, h = 0.0.5$

Step 1 $\frac{dy}{dx} = x^2 + 4x - \frac{1}{2}y$ So $f(x,y) = x^2 + 4x - \frac{1}{2}y$

$$y_{i+1} = y_i + hf(x_i, y_i)$$

$$y_1 = y_0 + hf(x_0, y_0)$$

$$y_1 = 4 + 0.05\left(0 + 4*0 - \frac{1}{2}*4\right) = 3.9 \qquad x_1 = x_0 + h \qquad x_1 = 0 + 0.05 = 0.05$$

Modified Euler method

$$y_1 = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)] \qquad y_{i+1} = y_i + \frac{h}{2}[f(x_i, y_i) + f(x_{i+1}, y_{i+1})]$$

$$y_1 = 4 + \frac{0.05}{2}\left[(-\frac{1}{2}*4) + (0.05)^2 + 4*(0.05) - \frac{1}{2}*3.9\right] = 3.906$$

Step 2

$$y_2 = y_1 + hf(x_1, y_1)$$

$$y_2 = 3.906 + 0.05\left[(0.05)^2 + 4*(0.05) - \frac{1}{2}*3.906\right] = 3.912$$

$$x_2 = x_1 + h$$

$$x_2 = 0.05 + 0.05 = 0.1$$

Modified Euler method

$$y_2 = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2)]$$

$$y_2 = 3.906 + \frac{0.05}{2}\left[(0.05) + 4*0.05 - \frac{1}{2}*3.906 + (0.1)^2 + 4*(0.1) - \frac{1}{2}*3.912\right] = \mathbf{3.868}$$

What is the Runge-Kutta 2nd order method?

The Runge-Kutta 2nd order method is a numerical technique used to solve an ordinary differential equation of the form

$$\frac{dy}{dx} = f(x, y), y(0) = y_0$$

Only first order ordinary differential equations can be solved by using the Runge-Kutta 2nd order method. In other sections, we will discuss how the Euler and Runge-Kutta methods are used to solve higher order ordinary differential equations or coupled (simultaneous) differential equations.

Runge-Kutta 2nd order method

To understand the Runge-Kutta 2nd order method, we need to derive Euler's method from the Taylor series.

$$y_{i+1} = y_i + \left.\frac{dy}{dx}\right|_{x_i, y_i} (x_{i+1} - x_i) + \frac{1}{2!}\left.\frac{d^2y}{dx^2}\right|_{x_i, y_i} (x_{i+1} - x_i)^2 + \frac{1}{3!}\left.\frac{d^3y}{dx^3}\right|_{x_i, y_i} (x_{i+1} - x_i)^3 + \ldots$$

$$= y_i + f(x_i, y_i)(x_{i+1} - x_i) + \frac{1}{2!} f'(x_i, y_i)(x_{i+1} - x_i)^2 + \frac{1}{3!} f''(x_i, y_i)(x_{i+1} - x_i)^3 + \ldots$$

As you can se the first two terms of the Taylor series

$$y_{i+1} = y_i + f(x_i, y_i)h$$

are Euler's method and hence can be considered to be the Runge-Kutta 1st order method.

The true error in the approximation is given by

$$E_t = \frac{f'(x_i, y_i)}{2!}h^2 + \frac{f''(x_i, y_i)}{3!}h^3 + \ldots$$

So what would a 2nd order method formula look like. It would include one more term of the Taylor series as follows.

$$y_{i+1} = y_i + f(x_i, y_i)h + \frac{1}{2!} f'(x_i, y_i)h^2$$

Let us take a generic example of a first order ordinary differential equation

$$\frac{dy}{dx} = e^{-2x} - 3y, y(0) = 5 \qquad f(x, y) = e^{-2x} - 3y$$

Now since *y* is a function of *x*,

$$f'(x, y) = \frac{\partial f(x, y)}{\partial x} + \frac{\partial f(x, y)}{\partial y}\frac{dy}{dx}$$

$$= \frac{\partial}{\partial x}\left(e^{-2x} - 3y\right) + \frac{\partial}{\partial y}\left[\left(e^{-2x} - 3y\right)\right]\left(e^{-2x} - 3y\right)$$

$$= -2e^{-2x} + (-3)\left(e^{-2x} - 3y\right) \qquad = -5e^{-2x} + 9y$$

The 2nd order formula for the above example would be

$$y_{i+1} = y_i + f(x_i, y_i)h + \frac{1}{2!} f'(x_i, y_i)h^2$$

$$= y_i + \left(e^{-2x_i} - 3y_i\right)h + \frac{1}{2!}\left(-5e^{-2x_i} + 9y_i\right)h^2$$

However, we already see the difficulty of having to in the above method. What Runge and Kutta did was write the 2nd order method as

$$y_{i+1} = y_i + (a_1 k_1 + a_2 k_2)h$$

where $k_1 = f(x_i, y_i)$ $\quad k_2 = f(x_i + p_1 h, y_i + q_{11} k_1 h)$

This form allows one to take advantage of the 2nd order method without having to calculate $f'(x, y)$

So how do we find the unknowns a_1, a_2, p_1, q_{11},

<u>Heun's method</u>

Here a_2=1/2 is chosen

$$a_1 = \frac{1}{2} \qquad p_1 = 1 \qquad q_{11} = 1$$

resulting in

$$y_{i+1} = y_i + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h$$

where

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + h, y_i + k_1 h)$$

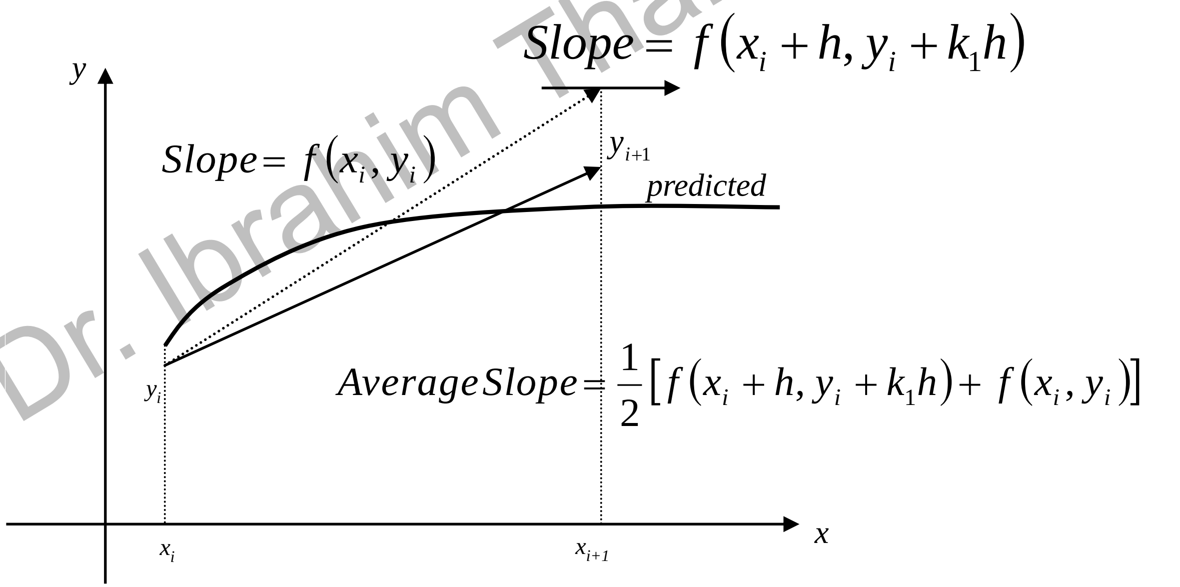

Figure 1 Runge-Kutta 2nd order method (Heun's method)

Midpoint Method

Here $a_2 = 1$ is chosen, giving

$$a_1 = 0$$

$$p_1 = \frac{1}{2}$$

$$q_{11} = \frac{1}{2}$$

resulting in

$$y_{i+1} = y_i + k_2 h$$

where

$$k_1 = f\left(x_i, y_i\right)$$

$$k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h\right)$$

Ralston's Method

Here $a_2 = \frac{2}{3}$ is chosen, giving

$$a_1 = \frac{1}{3}$$

$$p_1 = \frac{3}{4}$$

$$q_{11} = \frac{3}{4}$$

resulting in

$$y_{i+1} = y_i + \left(\frac{1}{3}k_1 + \frac{2}{3}k_2\right)h$$

where

$$k_1 = f\left(x_i, y_i\right)$$

$$k_2 = f\left(x_i + \frac{3}{4}h, y_i + \frac{3}{4}k_1 h\right)$$

Example

A ball at 1200K is allowed to cool down in air at an ambient temperature of 300K. Assuming heat is lost only due to radiation, the differential equation for the temperature of the ball is given by

$$\frac{d\theta}{dt} = -2.2067\times10^{-12}\left(\theta^4 - 81\times10^8\right), \theta(0) = 1200K$$

Find the temperature at at t= 480 seconds using Heun's method. Assume a step size of h= 240 seconds

Solution

$$\frac{d\theta}{dt} = -2.2067\times10^{-12}\left(\theta^4 - 81\times10^8\right) \qquad f(t,\theta) = -2.2067\times10^{-12}\left(\theta^4 - 81\times10^8\right)$$

$$\theta_{i+1} = \theta_i + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h \qquad k_1 = f(t_i, \theta_i) \qquad k_2 = f(t_i + h, \theta_i + k_1 h)$$

Step 1: $i = 0, t_0 = 0, \theta_0 = \theta(0) = 1200K$

$$\begin{aligned} k_1 &= f(t_0, \theta_o) \\ &= f(0, 1200) \\ &= -2.2067\times10^{-12}\left(1200^4 - 81\times10^8\right) \\ &= -4.5579 \end{aligned}$$

$$\begin{aligned} k_2 &= f(t_0 + h, \theta_0 + k_1 h) \\ &= f(0 + 240, 1200 + (-4.5579)240) \\ &= f(240, 106.09) \\ &= -2.2067\times10^{-12}\left(106.09^4 - 81\times10^8\right) \\ &= 0.017595 \end{aligned}$$

$$\begin{aligned} \theta_1 &= \theta_0 + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h \\ &= 1200 + \left(\frac{1}{2}(-4.5579) + \frac{1}{2}(0.017595)\right)240 \\ &= 1200 + (-2.2702)240 \\ &= 655.16K \end{aligned}$$

Step 2: $i = 1, t_1 = t_0 + h = 0 + 240 = 240, \theta_1 = 655.16K$

$$k_1 = f(t_1, \theta_1)$$
$$= f(240, 655.16)$$
$$= -2.2067 \times 10^{-12}(655.16^4 - 81 \times 10^8)$$
$$= -0.38869$$

$$k_2 = f(t_1 + h, \theta_1 + k_1 h)$$
$$= f(240 + 240, 655.16 + (-0.38869)240)$$
$$= f(480, 561.87)$$
$$= -2.2067 \times 10^{-12}(561.87^4 - 81 \times 10^8)$$
$$= -0.20206$$

$$\theta_2 = \theta_1 + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h$$
$$= 655.16 + \left(\frac{1}{2}(-0.38869) + \frac{1}{2}(-0.20206)\right)240$$
$$= 655.16 + (-0.29538)240$$
$$= 584.27K$$

The exact solution of the ordinary differential equation is given by the solution of a non-linear equation as

$$0.92593 \ln \frac{\theta - 300}{\theta + 300} - 1.8519 \tan^{-1}(0.0033333\theta) = -0.22067 \times 10^{-3} t - 2.9282$$

The solution to this nonlinear equation at t=480 seconds is

$$\theta(480) = 647.57K$$

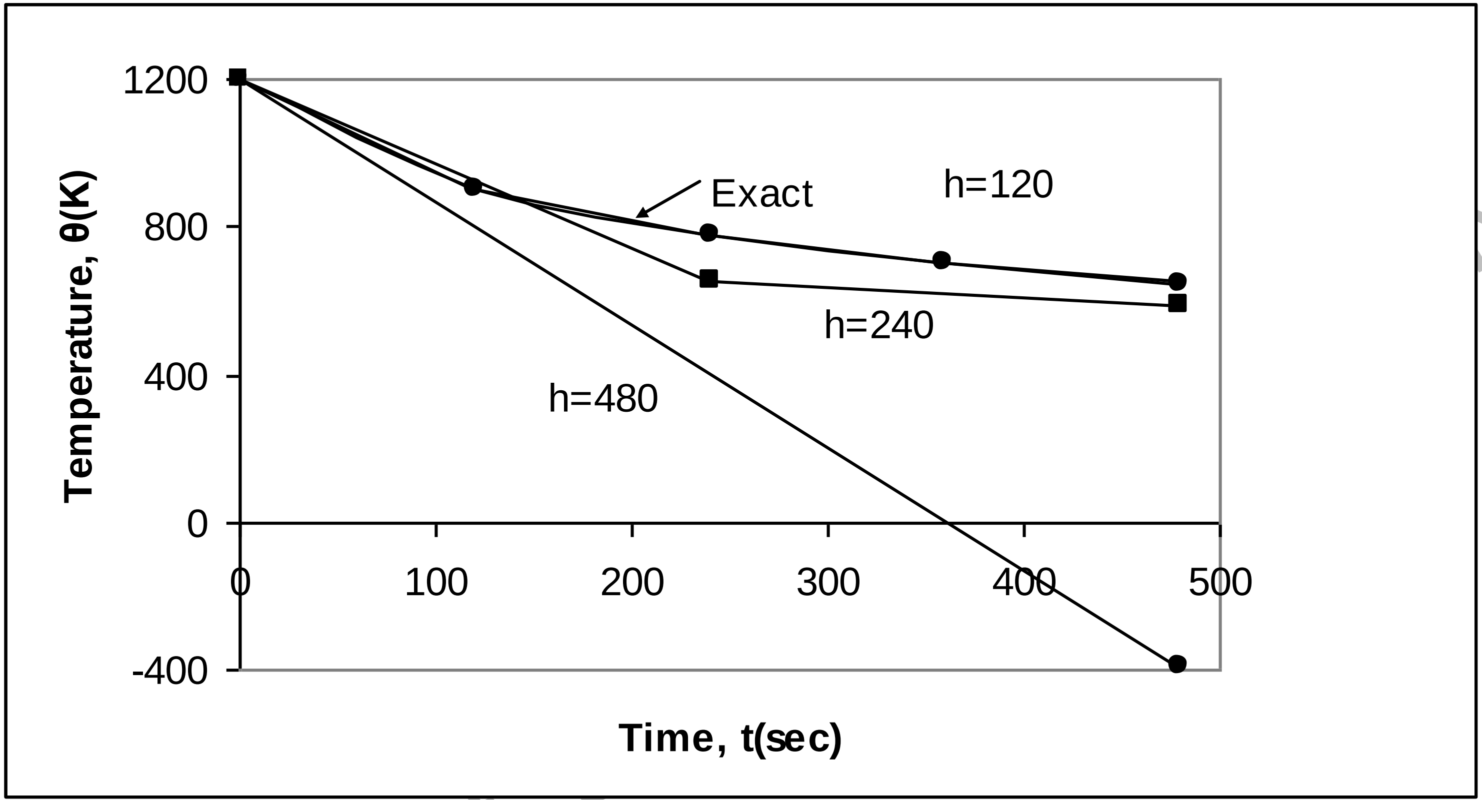

Figure Heun's method results for different step sizes

What is the Runge-Kutta 4th order method?

Runge-Kutta 4th order method is a numerical technique used to solve ordinary differential equation of the form

$$\frac{dy}{dx} = f(x, y), y(0) = y_0$$

So only first order ordinary differential equations can be solved by using the Runge-Kutta 4th order method. In other sections, we have discussed how Euler and Runge-Kutta methods are used to solve higher order ordinary differential equations or coupled (simultaneous) differential equations

The Runge-Kutta 4th order method is based on the following

$$y_{i+1} = y_i + \left(a_1 k_1 + a_2 k_2 + a_3 k_3 + a_4 k_4\right)h$$

where knowing the value of $y = y_i \ at \ x_i$ we can find the value of $y = y_{i+1} \ at \ x_{i+1}$ and $h = x_{i+1} - x_i$

Equation (1) i equated to the first five terms of Taylor series

$$y_{i+1} = y_i + \frac{dy}{dx}\Big|_{x_i, y_i}(x_{i+1} - x_i) + \frac{1}{2!}\frac{d^2y}{dx^2}\Big|_{x_i, y_i}(x_{i+1} - x_i)^2 + \frac{1}{3!}\frac{d^3y}{dx^3}\Big|_{x_i, y_i}(x_{i+1} - x_i)^3$$

$$+ \frac{1}{4!}\frac{d^4y}{dx^4}\Big|_{x_i, y_i}(x_{i+1} - x_i)^4$$

Knowing that $\frac{dy}{dx} = f(x, y)$ and $x_{i+1} - x_i = h$

Runge-Kutta 4th Order Method

$$y_{i+1} = y_i + f(x_i, y_i)h + \frac{1}{2!} f^{'}(x_i, y_i)h^2 + \frac{1}{3!} f^{''}(x_i, y_i)h^3 + \frac{1}{4!} f^{'''}(x_i, y_i)h^4$$

Based on equating previous equations , one of the popular solutions used is

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$

where

$$k_1 = f(x_i, y_i)$$

$$k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h \right)$$

$$k_3 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_2 h \right)$$

$$k_4 = f(x_i + h, y_i + k_3 h)$$

Example

A ball at 1200K is allowed to cool down in air at an ambient temperature of 300K. Assuming heat is lost only due to radiation, the differential equation for the temperature of the ball is given by

$$\frac{d\theta}{dt} = -2.2067\times10^{-12}\left(\theta^4 - 81\times10^8\right), \theta(0) = 1200K$$

Find the temperature at t =480 seconds using Runge-Kutta 4th order method. Assume a step size of h= 240 seconds

$$\frac{d\theta}{dt} = -2.2067\times10^{-12}\left(\theta^4 - 81\times10^8\right)$$

$$f(t,\theta) = -2.2067\times10^{-12}\left(\theta^4 - 81\times10^8\right)$$

$$\theta_{i+1} = \theta_i + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)h$$

Step 1: $i = 0, t_0 = 0, \theta_0 = \theta(0) = 1200$

$$k_1 = f(t_0, \theta_o) = f(0,1200) = -2.2067\times10^{-12}\left(1200^4 - 81\times10^8\right) = -4.5579$$

$$k_2 = f\left(t_0 + \frac{1}{2}h, \theta_0 + \frac{1}{2}k_1 h\right) = f\left(0 + \frac{1}{2}(240), 1200 + \frac{1}{2}(-4.5579)240\right)$$

$$= f(120, 653.05) = -2.2067\times10^{-12}\left(653.05^4 - 81\times10^8\right) = -0.38347$$

$$k_3 = f\left(t_0 + \frac{1}{2}h, \theta_0 + \frac{1}{2}k_2 h\right) = f\left(0 + \frac{1}{2}(240), 1200 + \frac{1}{2}(-0.38347)240\right)$$

$$= f(120, 1154.0) = 2.2067 \times 10^{-12}\left(1154.0^4 - 81 \times 10^8\right) = -3.8954$$

$$k_4 = f(t_0 + h, \theta_0 + k_3 h) = f(0 + (240), 1200 + (-3.984)240)$$

$$= f(240, 265.10) = 2.2067 \times 10^{-12}\left(265.10^4 - 81 \times 10^8\right) = 0.0069750$$

$$\theta_1 = \theta_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$

$$= 1200 + \frac{1}{6}(-4.5579 + 2(-0.38347) + 2(-3.8954) + (0.069750))240$$

$$= 1200 + (-2.1848) \times 240 \quad = 675.65\text{K}$$

θ_1 is the approximate temperature at $t = t_1 \quad = t_0 + h \; = 0 + 240 = 240$

$$\theta_1 = \theta(240) \approx 675.65\text{K}$$

For $i = 1, t_1 = 240, \theta_1 = 675.65\text{K}$

$$k_1 = f(t_1, \theta_1) \quad = f(240, 675.65)$$

$$= -2.2067 \times 10^{-12}\left(675.65^4 - 81 \times 10^8\right) \quad = -0.44199$$

$$k_2 = f\left(t_1 + \frac{1}{2}h, \theta_1 + \frac{1}{2}k_1 h\right) = f\left(240 + \frac{1}{2}(240), 675.65 + \frac{1}{2}(-0.44199)240\right)$$

$$= f(360, 622.61)$$

$$= -2.2067\times10^{-12}\left(622.61^4 - 81\times10^8\right) = -0.31372$$

$$k_3 = f\left(t_1 + \frac{1}{2}h, \theta_1 + \frac{1}{2}k_2 h\right)$$

$$= f\left(240 + \frac{1}{2}(240), 675.65 + \frac{1}{2}(-0.31372)\times240\right)$$

$$= f(360, 638.00)$$

$$= -2.2067\times10^{-12}\left(638.00^4 - 81\times10^8\right) = -0.34775$$

$$k_4 = f(t_1 + h, \theta_1 + k_3 h)$$

$$= f(240 + 240, 675.65 + (-0.34775)\times240)$$

$$= f(480, 592.19)$$

$$= 2.2067\times10^{-12}\left(592.19^4 - 81\times10^8\right) = -0.25351$$

$$\theta_2 = \theta_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$

$$= 675.65 + \frac{1}{6}(-0.44199 + 2(-0.31372) + 2(-0.34775) + (-0.25351))240$$

$$= 675.65 + \frac{1}{6}(-2.0184)240$$

$$= 594.91K$$

q_2 is the approximate temperature at

$$t_2 = t_1 + h = 240 + 240 = 480$$

$$\theta(480) \approx \theta_2 = 594.91K$$

The exact solution of the ordinary differential equation is given by the solution of a non-linear equation as

$$0.92593\ln\frac{\theta - 300}{\theta + 300} - 1.8519\tan^{-1}(0.00333\theta) = -0.22067\times10^{-3}t - 2.9282$$

The solution to this nonlinear equation at t=480 seconds is $\theta(480) = 647.57K$

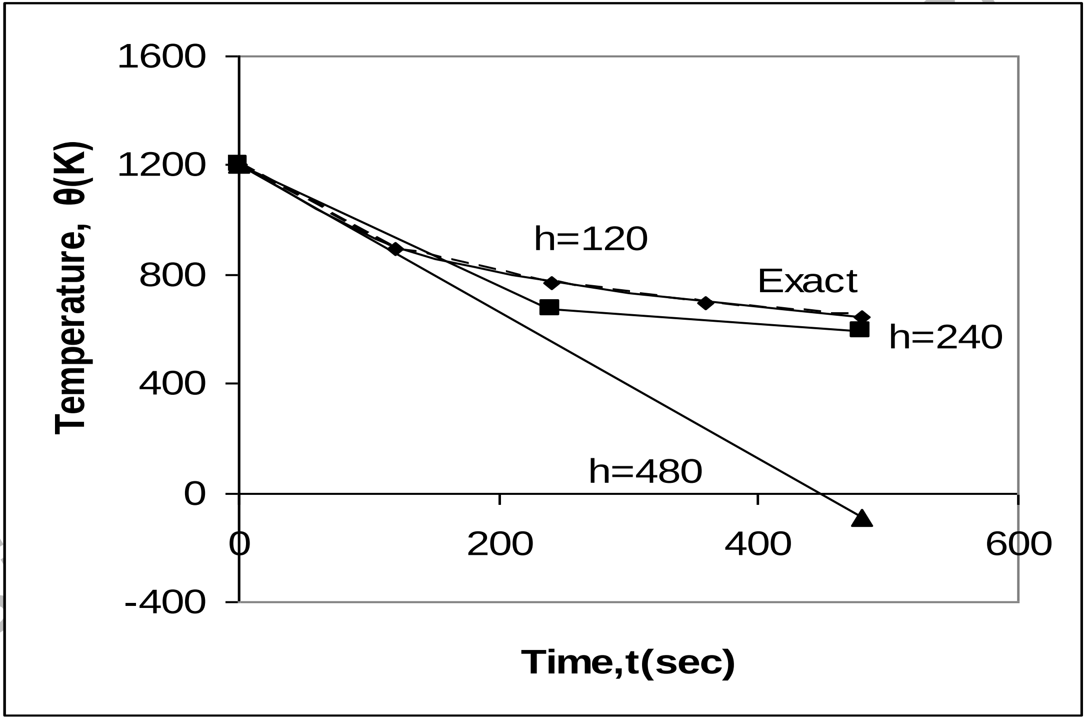

Figure. Comparison of Runge-Kutta 4th order method with exact solution